

A SIMPLE PROOF OF THE LEBESGUE DECOMPOSITION THEOREM

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The aim of this short note is to present an elementary, self-contained, and direct proof for the classical Lebesgue decomposition theorem. In fact, I will show that the absolutely continuous part just measures the squared semidistance of the characteristic functions from a suitable subspace.

This approach also gives a decomposition in the finitely additive case, but it differs from the Lebesgue-Darst decomposition [1], because the involved absolute continuity concepts are different.

Notations. Let \mathcal{A} be a σ -algebra over $X \neq \emptyset$, and consider the finite measures $\mu, \nu : \mathcal{A} \rightarrow \mathbb{R}_+$ on it. The measure μ is ν -absolutely continuous ($\mu \ll \nu$, in symbols) if $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathcal{A}$. Singularity of μ and ν (denoted by $\mu \perp \nu$) means that the only measure dominated by both μ and ν is the zero measure. As it is known, this is equivalent with the existence of a measurable set $P \in \mathcal{A}$ such that $\mu(P) = \nu(X \setminus P) = 0$.

Theorem. *Let μ and ν be finite measures on \mathcal{A} . Then μ splits uniquely into $\mu_{ac} \ll \nu$ and $\mu_s \perp \nu$.*

Proof. Consider the real vector space \mathcal{E} of real valued \mathcal{A} -measurable step-functions and let \mathcal{N} be the linear subspace generated by the characteristic functions of those measurable sets A such that $\nu(A) = 0$. Define the set function μ_{ac} by

$$\mu_{ac}(A) := \inf_{\psi \in \mathcal{N}} \int_X |\mathbb{1}_A - \psi|^2 d\mu \quad (A \in \mathcal{A}).$$

It is clear that $\mu_{ac} \leq \mu$ ($\psi := \mathbb{1}_\emptyset$), and that $\nu(A) = 0$ implies $\mu_{ac}(A) = 0$ ($\psi := \mathbb{1}_A$). Furthermore, trivial verification shows that if A and B are disjoint elements of \mathcal{A} , then

$$\inf_{\psi \in \mathcal{N}} \int_X |\mathbb{1}_{A \cup B} - \psi|^2 d\mu = \inf_{\psi \in \mathcal{N}} \int_X |\mathbb{1}_A - \psi|^2 d\mu + \inf_{\psi \in \mathcal{N}} \int_X |\mathbb{1}_B - \psi|^2 d\mu.$$

Since μ_{ac} is nonnegative, additive, and dominated by the measure μ , we infer that μ_{ac} is a measure itself.

What is left is to show that $\mu_s := \mu - \mu_{ac}$ and ν are singular, and that the decomposition is unique. Both follow immediately from the fact that μ_{ac} is maximal among those measures ϑ such that $\vartheta \leq \mu$ and $\vartheta \ll \nu$. Indeed, let ϑ be such a measure, $\psi \in \mathcal{N}$, and observe that

$$\vartheta(A) = \int_X |\mathbb{1}_A|^2 d\vartheta = \int_X |\mathbb{1}_A - \psi|^2 d\vartheta \leq \int_X |\mathbb{1}_A - \psi|^2 d\mu.$$

Taking the infimum over \mathcal{N} we obtain that $\vartheta \leq \mu_{ac}$.

Now, let η be a measure, such that $\eta \leq \nu$ and $\eta \leq \mu - \mu_{ac}$. In this case, $\mu_{ac} + \eta \leq \mu$ and $\mu_{ac} + \eta \ll \nu$, thus $\eta = 0$. If $\mu = \mu_1 + \mu_2$, where $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$, then $\mu_{ac} - \mu_1$ is a measure, which is simultaneously ν -absolutely continuous and ν -singular. This yields that $\mu_1 = \mu_{ac}$. \square

REFERENCES

- [1] Tarsay, Zs., *A functional analytic proof of the Lebesgue-Darst decomposition theorem*, Real Analysis Exchange, Vol. 39(1), 2013/2014, 241–248.

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